

Local periods for discrete series representations

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Abstract

Let (G, H) be a symmetric pair over a p -adic field and π a discrete series representation of G . In this paper, for some type of symmetric pairs (G, H) , we show that local periods in $\text{Hom}_H(\pi, \mathbb{C})$ can be constructed by integrating the matrix coefficients of π over H .

1 Introduction

Let F be a non-archimedean local field of characteristic 0. Let \mathbf{G} be a connected reductive group over F and \mathbf{H} a unimodular spherical subgroup of \mathbf{G} , which means that $\mathbf{X} = \mathbf{H} \backslash \mathbf{G}$ is a spherical variety. Write $G = \mathbf{G}(F)$ and $H = \mathbf{H}(F)$.

Let π be a unitary irreducible admissible representation of G and V_π the space of π . We say that π is H -distinguished if the space $\text{Hom}_H(\pi, \mathbb{C})$ is nonzero. We call elements of $\text{Hom}_H(\pi, \mathbb{C})$ local periods. If π is H -distinguished, how to explicitly construct nonzero local periods is an important question. This is part of the local theory of automorphic periods.

A natural way to construct local periods is to consider the integration of the matrix coefficients of π over H . More precisely, let \mathbf{Z} be the split component of the center of \mathbf{G} and $\mathbf{Z}_\mathbf{H} = \mathbf{Z} \cap \mathbf{H}$. Write $Z_H = \mathbf{Z}_\mathbf{H}(F)$. Note that if π is H -distinguished then the restriction of the central character of π to Z_H is trivial. Fix a G -invariant hermitian inner product $\langle \cdot, \cdot \rangle$ on V_π . We formally define a pairing \mathcal{L} on $V_\pi \times V_\pi$ by

$$\mathcal{L}(v, u) = \int_{H/Z_H} \langle \pi(h)v, u \rangle \, dh. \quad (1)$$

Note that if \mathcal{L} is well defined then it is bi- H -invariant and the map \mathcal{L}_u given by

$$\mathcal{L}_u : v \mapsto \mathcal{L}(v, u), \quad v \in V_\pi \quad (2)$$

belongs to $\text{Hom}_H(\pi, \mathbb{C})$. If \mathcal{L} is well defined, we denote by

$$\mathcal{H}(\pi) = \{\mathcal{L}_u\}_{u \in V_\pi}$$

the subspace of $\text{Hom}_H(\pi, \mathbb{C})$. Then two natural questions arise:

1. whether \mathcal{L} is well defined;
2. if \mathcal{L} is well defined, whether we have $\mathcal{H}(\pi) = \text{Hom}_H(\pi, \mathbb{C})$.

In this paper, we restrict ourselves to the following situations:

- either \mathbf{X} is a *symmetric space* and π is a discrete series representation;

- or π is supercuspidal and \mathbf{X} is a symmetric space or *wavefront* spherical variety.

We always assume that \mathbf{G} is split when we require \mathbf{X} to be a wavefront spherical variety. We refer to [SV12, §2.1] for the definition of wavefront spherical varieties.

When π is supercuspidal, \mathcal{L} is always well defined. The following notions are natural when we consider discrete series representations.

Definition 1.1. A discrete series representation π of G is called *H-integrable* if all its matrix coefficients lie in $L^1(H/Z_H)$, i.e. \mathcal{L} is well defined.

Definition 1.2. A symmetric space $\mathbf{X} = \mathbf{H} \backslash \mathbf{G}$ is called *strongly discrete* if all discrete series representations of G are H -integrable.

However, in this paper, we need the following notion of very strongly discreteness which is stronger than the notion of strongly discreteness.

Definition 1.3. A symmetric space is called *very strongly discrete* if the linear form

$$\mathcal{L} : \mathcal{C}(G/Z) \longrightarrow \mathbb{C}, \quad f \mapsto \int_{H/Z_H} f(h) \, dh$$

is well defined and continuous, where $\mathcal{C}(G/Z)$ is the Schwartz-Harish-Chandra space of G/Z .

Our main theorems are:

Theorem 1.4. Suppose that \mathbf{X} is a very strongly discrete symmetric space and π is a discrete series representation. Then $\mathcal{H}(\pi) = \text{Hom}_H(\pi, \mathbb{C})$.

Theorem 1.5. Suppose that \mathbf{X} is a symmetric space or a wavefront spherical variety and π is a supercuspidal representation. Then $\mathcal{H}(\pi) = \text{Hom}_H(\pi, \mathbb{C})$.

Remark 1.6. Theorem 1.4 shows that any local period in $\text{Hom}_H(\pi, \mathbb{C})$ is given by integrating some matrix coefficients over H , which is an analog of the global automorphic period. This theorem can be used to study the factorization of the global periods into local ones. We refer to [SV12, §17] for a further discussion on the link between the global and local theory.

Remark 1.7. When \mathbf{X} is a symmetric space and π is of the form $\text{ind}_J^G \kappa$ for some open compact subgroup J of G and some irreducible smooth representation κ of J , results of the above kind have been obtained by Hakim, Mao and Murnaghan. See [Mur08, §8] for a survey. Our method is different from theirs.

Remark 1.8. When \mathbf{X} is a strongly tempered spherical variety (cf. [SV12, §6.2] for the definition) and π is a tempered representation, the pairing \mathcal{L} is well defined by the definition. In this case, Sakellaridis and Venkatesh [SV12, Theorem 6.4.1] showed that $\text{Hom}_H(\pi, \mathbb{C})$ is nonzero if and only if \mathcal{L} is nonzero, by using the Plancherel decomposition of $L^2(X)$ with $X = H \backslash G$. When $(\mathbf{G}, \mathbf{H}) = (\text{SO}_n \times \text{SO}_{n+1}, \text{SO}_n)$ is in the setting of Gan-Gross-Prasad conjecture for special orthogonal groups, Ichino and Ikeda [II10, Proposition 1.1] showed that \mathbf{X} is strongly tempered; Waldspurger [Wal12b, Théorème 1] showed that $\text{Hom}_H(\pi, \mathbb{C}) \leq 1$ (basing on the method of [AGRS10]) and [Wal12a, Proposition 5.6] also proved that $\text{Hom}_H(\pi, \mathbb{C})$ is nonzero if and only if \mathcal{L} is nonzero by a

different method from that of [SV12]. Thus, in this case, $\mathcal{H}(\pi) = \text{Hom}_H(\pi, \mathbb{C})$. When $(\mathbf{G}, \mathbf{H}) = (\text{U}_n \times \text{U}_{n+1}, \text{U}_n)$ is in the setting of Gan-Gross-Prasad conjecture for unitary groups, analogous result was proved by Beuzart-Plessis ([BP12, Theorem 14.3.1] and [BP15, Theorem 8.4.1]) for both archimedean and non-archimedean cases.

Remark 1.9. In general, \mathcal{L} is not well defined. For example, if $(\mathbf{G}, \mathbf{H}) = (\text{GL}_{2n}, \text{Sp}_{2n})$, the matrix coefficients of a discrete series representation may not belong to $L^1(H/Z_H)$.

Remark 1.10. In an earlier version of this paper, we only considered the case when π is supercuspidal. At the same time, we began to study the case when π is discrete series. However, at that time, we could only show some special cases of symmetric spaces are very strongly discrete and did not prove Theorem 1.4. Recently, we realize how to prove Theorem 1.4 when \mathbf{X} is very strongly discrete. At the same time, Gurevich-Offen [GO15] give a criterion for strongly discreteness (cf. [GO15, Theorem 4.4]) and also a sufficient condition for strongly discreteness (cf. [GO15, Corollary 5.4]). We can show that the condition in [GO15, Corollary 5.4] is sufficient and necessary for very strongly discreteness.

As a consequence, under the same assumption in Theorem 1.4 or Theorem 1.5, we have the following expression (Corollary 1.11) for the spherical character $\Phi_{\pi, \ell}$ associated to $\ell \in \text{Hom}_H(\pi, \mathbb{C})$. Recall that, for $\ell \in \text{Hom}_H(\pi, \mathbb{C})$, the spherical character $\Phi_{\pi, \ell}$ is defined to be the distribution on G given by

$$\Phi_{\pi, \ell}(f) := \sum_{v \in \text{ob}(\pi)} \ell(\pi(f)v) \overline{\ell(v)}, \quad f \in C_c^\infty(G),$$

where $\text{ob}(\pi)$ is an orthonormal basis of V_π . By Theorem 1.4 or Theorem 1.5, there exists $v_0 \in V_\pi$ such that $\ell = \mathcal{L}_{v_0}$. The corollary below is analogous to [Mur08, Theorem 6.1] and [IZ14, Lemma A.3]. For the proof, see that of [IZ14, Lemma A.3].

Corollary 1.11. *For all $f \in C_c^\infty(G)$, we have*

$$\Phi_{\pi, \ell}(f) = \int_{H/H \cap Z} \int_{H/H \cap Z} \left(\int_G f(g) \phi(h_2 g h_1) \, dg \right) \, dh_1 \, dh_2,$$

where

$$\phi(g) = \langle \pi(g)v_0, v_0 \rangle, \quad g \in G.$$

Remark 1.12. Combined with other inputs, Corollary 1.11 can be used to study the supports of spherical characters, which has potential applications in simple relative trace formula. For example, see [IZ14, Theorem A.2], [FMW13, Proposition 4.5] and [Zha15, Theorem 1.2].

The rest of the paper is organized as follows. In §2, some basic notions and properties that will be used in the paper are recalled. In §3, we study the very strongly discrete symmetric spaces. Some aspects of Gurevich-Offen's work will be briefly reviewed there. In order to prove Theorem 1.5, we also recall a property of supercuspidal representation (see Lemma 3.1). In §4, we prove Theorem 1.4. Our proof is motivated by those of [Wal12a, Proposition 5.6] and [BP15, Theorem 8.4.1]. The proof of Theorem 1.5 is the same but much more easier. We leave it to the reader.

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2 Notations and preliminaries

Estimates Let S be a set. If f_1 and f_2 are positive functions on S , we write $f_1 \prec f_2$ if there exists $c > 0$ such that $f_1(s) \leq cf_2(s)$ for all $s \in S$; we write $f_1 \asymp f_2$ if both $f_1 \prec f_2$ and $f_2 \prec f_1$.

Fields Let F be a non-archimedean local field of characteristic 0. Denote by $|\cdot|_F$ the normalized absolute value of F , and by \mathfrak{o}_F the ring of integers of F .

Groups All the algebraic groups mentioned in the paper are defined over F . We use boldface letter to denote an algebraic group, and use the corresponding nonbold letter to denote the associated group of F -rational points. For an algebraic variety \mathbf{X} over F , $X = \mathbf{X}(F)$ is equipped with the natural topology induced from F , which is a locally compact totally disconnected topological space.

For a connected reductive group \mathbf{G} , we denote by \mathfrak{a}_G the real vector space $\text{Hom}_{\mathbb{Z}}(X^*(\mathbf{G}), \mathbb{R})$ where $X^*(\mathbf{G}) = \text{Hom}_F(\mathbf{G}, \mathbb{G}_m)$ is the group of rational characters of \mathbf{G} . We have the Harish-Chandra map $H_G : G \rightarrow \mathfrak{a}_G$ defined by

$$\langle \chi, H_G(g) \rangle = \log(|\chi(g)|_F), \quad g \in G, \chi \in X^*(\mathbf{G}).$$

Let \mathbf{A}_0 be a maximal split torus of \mathbf{G} and $\mathbf{P}_0 = \mathbf{M}_0 \mathbf{U}_0$ a minimal parabolic subgroup such that $\mathbf{A}_0 \subset \mathbf{M}_0$ where \mathbf{U}_0 is the unipotent radical of \mathbf{P}_0 and \mathbf{M}_0 is the Levi subgroup. There is a canonical identification

$$\mathfrak{a}_0 := \mathfrak{a}_{A_0} \simeq \mathfrak{a}_{M_0}.$$

Denote $H_0 = H_{M_0}$ for short. Let $\Delta(A_0, P_0)$ be the set of simple roots of \mathbf{A}_0 in the Lie algebra of \mathbf{P}_0 . Set

$$M_0^+ = \{m \in M_0; \langle \alpha, H_0(m) \rangle \geq 0, \forall \alpha \in \Delta(A_0, P_0)\}.$$

Let K be an A_0 -good maximal compact subgroup of G and M_0^1 the kernel of H_{M_0} . Then we have a Cartan decomposition

$$G = \bigsqcup_{m \in M_0^+ / M_0^1} KmK.$$

Recall that (cf. [Wal03, §I.1, (5)])

$$\text{vol}(KmK) \asymp \delta_0^{-1}(m) \tag{3}$$

as functions on M_0^+ , where $\delta_0 = \delta_{P_0}$ is the modular character of P_0 . What we actually use is the following Cartan decomposition (cf. [Ren10, Theorem V 3.21]). Let $A_0^+ = A_0 \cap M_0^+$ and $A_0^1 = A_0 \cap M_0^1$, then there exists a finite subset F_0 of M_0^+ such that

$$G = \bigsqcup_{a \in A_0^+ / A_0^1} \bigsqcup_{c \in F_0} KacK. \tag{4}$$

Some functions on G Let \mathbf{G} be a connected reductive group and keep the notations as before. Fix an algebraic embedding $\tau : \mathbf{G} \rightarrow \mathrm{GL}_n$ over F . We can and do assume that $K \subset \mathrm{GL}_n(\mathfrak{o}_F)$. Then a norm function $\|\cdot\|$ on G is given by

$$\|g\| := \sup_{i,j} \sup (|\tau(g)_{ij}|_F, |\tau(g^{-1})_{ij}|_F).$$

Set $\sigma(g) = \log \|g\|$ to be the log-norm function on G . We refer to [Wal03, §I.1] for the properties of σ . Especially, fixing a W^G -invariant norm $|\cdot|$ on \mathfrak{a}_0 , where W^G is the Weyl group of A_0 in G , we have

$$1 + \sigma(m) \asymp 1 + |H_0(m)|$$

as functions on M_0 . Also, for a compact subset ω of G , we have

$$\sup_{\gamma_1, \gamma_2 \in \omega} 1 + \sigma(\gamma_1 g \gamma_2) \asymp 1 + \sigma(g)$$

as functions on G . For an algebraic (affine) variety \mathbf{X} over F , there is a general notion of norms on X . We refer to [Kot05, §18] for the precise definitions and some important properties of norms.

Let $\Xi(g)$ be the Harish-Chandra function of G given by $\Xi(g) = \langle \pi(g)v_0, v_0 \rangle$ where v_0 is the unique K -invariant element of $\mathrm{Ind}_{P_0}^G \mathbf{1}$ such that $v_0(1) = 1$. Recall that Ξ is positive real-valued, bi- K -invariant and there exists $d \in \mathbb{N}$ such that

$$\delta_0^{1/2}(m) \prec \Xi(m) \prec \delta_0^{1/2}(m)(1 + \sigma(m))^d \quad (5)$$

as functions on M_0^+ (cf. [Wal03, Lemme II.1.1]). Also recall that for any $g_1, g_2 \in G$ we have (cf. [Wal03, Lemme II.1.3])

$$\int_K \Xi(g_1 k g_2) dk = \Xi(g_1) \Xi(g_2). \quad (6)$$

For a compact subset ω of G , we have (cf. [Sil79, Lemma 4.2.3])

$$\sup_{\gamma_1, \gamma_2 \in \omega} \Xi(\gamma_1 g \gamma_2) \asymp \Xi(g) \quad (7)$$

as functions on G .

We denote by $\mathcal{C}(G)$ the Schwartz-Harish-Chandra space of G , which is the space of bi- J -invariant continuous functions f on G for some open compact subgroup J of G such that for each $r \in \mathbb{R}$

$$|f(g)| \prec \Xi(g)(1 + \sigma(g))^{-r} \quad (8)$$

as functions on G . The space $\mathcal{C}(G)$ is a locally convex and compact topological vector space. We refer to [Wal03, §II.1] for the precise description of the topology on $\mathcal{C}(G)$.

Symmetric spaces Let \mathbf{G} be a connected reductive group and θ a rational involution of \mathbf{G} defined over F . Let \mathbf{G}^θ be the group of the fixed points of θ and $(\mathbf{G}^\theta)^0$ be the connected component of \mathbf{G}^θ containing the identity. Let \mathbf{H} be the subgroup of \mathbf{G} such that $(\mathbf{G}^\theta)^0 \subset \mathbf{H} \subset \mathbf{G}^\theta$. Then (\mathbf{G}, \mathbf{H}) is called a symmetric pair and the geometric quotient $\mathbf{X} = \mathbf{H} \backslash \mathbf{G}$ is called a symmetric space.

A split torus \mathbf{A} of \mathbf{G} is called θ -split if $\theta(a) = a^{-1}$ for any $a \in \mathbf{A}$. A parabolic subgroup \mathbf{P} of \mathbf{G} is called a θ -parabolic subgroup if \mathbf{P} and $\theta(\mathbf{P})$ are opposite parabolic subgroups. In such a case, we always take $\mathbf{M} = \mathbf{P} \cap \theta(\mathbf{P})$ for a Levi subgroup of \mathbf{P} , which is θ -stable. It is known that HP is open in G when \mathbf{P} is a θ -parabolic subgroup. Let $\mathbf{P} = \mathbf{M}\mathbf{U}$ be a θ -parabolic of \mathbf{G} and $\mathbf{A}_{\mathbf{P},\theta}$ the maximal θ -split torus of the center of \mathbf{M} . Denote by $\Delta(A_{P,\theta}, P)$ the set of simple roots of $\mathbf{A}_{\mathbf{P},\theta}$ in the Lie algebra of \mathbf{P} and set

$$A_{P,\theta}^+ = \{a \in A_{P,\theta}; |\alpha(a)|_F \leq 1 \ \forall \alpha \in \Delta(A_{P,\theta}, P)\}.$$

We have the relative Cartan decomposition for symmetric spaces (cf. [BO07, Theorem 1.1]): there exists a compact subset Ω of G and a finite set \mathcal{P} of minimal θ -parabolic subgroups of \mathbf{G} such that

$$G = \bigcup_{\mathbf{P} \in \mathcal{P}} HA_{P,\theta}^+ \Omega. \quad (9)$$

Some functions on $H \backslash G$ Let (\mathbf{G}, \mathbf{H}) be a symmetric pair. We recall some functions on $H \backslash G$ introduced in [Lag08, §3] and their basic properties. Consider the symmetric map $s : H \backslash G \rightarrow G$ given by $s(g) = \theta(g^{-1})g$. The following functions on $H \backslash G$ are all defined by the pullback of some functions on G via s . Set

$$\Theta := (s^* \Xi)^{\frac{1}{2}}, \quad N_d := (1 + s^* \sigma)^d$$

for $d \in \mathbb{Z}$, that is,

$$\Theta(Hg) = \Xi(s(g))^{\frac{1}{2}}, \quad N_d(Hg) = (1 + \sigma(s(g)))^d$$

for $Hg \in H \backslash G$. We denote $N = N_1$ for short.

Let $\mathbf{P} = \mathbf{M}\mathbf{U}$ be a minimal θ -parabolic subgroup of \mathbf{G} and ω a compact subset of G . Choose a norm on $|\cdot|$ on \mathfrak{a}_M . Then, as functions on $A_{P,\theta}$, we have (cf. [Lag08, Lemma 7])

$$\sup_{\gamma \in \omega} N(Ha\gamma) \asymp 1 + |H_M(a)|. \quad (10)$$

Also, as functions on $A_{P,\theta}$, there exists $d, d' \in \mathbb{N}$ such that

$$\delta_P^{\frac{1}{2}}(a)N_{-d}(Ha) \prec \sup_{\gamma \in \omega} \Theta(Ha\gamma) \prec \delta_P^{\frac{1}{2}}(a)N_{d'}(Ha) \quad (11)$$

(cf. [Lag08, Proposition 6]).

We denote by $\mathcal{C}(H \backslash G)$ the Schwartz-Harish-Chandra space of $H \backslash G$ (introduced in [DH14, Definition 4.1]), which is the space of right- J -invariant functions f on $H \backslash G$ for some compact open subgroup J of G such that for any $d \in \mathbb{N}$

$$|f(x)| \prec \Theta(x)N_{-d}(x) \quad (12)$$

as functions on $H \backslash G$.

Representations Let \mathbf{G} be a connected reductive group and \mathbf{Z} the split component of the center of G . Let (π, V_π) be an irreducible admissible representation. The functions $\phi(g) = \langle \pi(g)v, u \rangle$ on G for some $v, u \in V_\pi$ are called matrix coefficients of π . Recall that π is called supercuspidal if its matrix coefficients are compactly supported modulo Z , and π is called discrete series if it is unitary and its matrix coefficients are square-integrable on G/Z . It is known that if π is discrete series then the absolute values of all its matrix coefficients belong to the Schwartz-Harish-Chandra space $\mathcal{C}(G/Z)$.

Now let (\mathbf{G}, \mathbf{H}) be a symmetric pair. For $\ell \in \text{Hom}_H(\pi, \mathbb{C})$ and $v \in V_\pi$, the generalized matrix coefficient of ℓ associated to v is defined by

$$\varphi_{\ell, v}(g) = \ell(\pi(g)v), \quad g \in G,$$

which is a continuous function on $H \backslash G$.

We say that π is relatively supercuspidal if its generalized matrix coefficients $\varphi_{\ell, v}$ are compactly supported modulo ZH for all $\ell \in \text{Hom}_H(\pi, \mathbb{C})$ and $v \in V_\pi$. It is known that if π is supercuspidal then it is relatively supercuspidal (cf. [KT08, Proposition 8.1]).

We say that π is a relatively discrete series representation if π is unitary and

$$\int_{ZH \backslash G} |\varphi_{\ell, v}(g)|^2 dg < \infty$$

for all $\ell \in \text{Hom}_H(\pi, \mathbb{C})$ and all $v \in V_\pi$. It is known that if π is a discrete series representation then it is relatively discrete series (cf. [KT10, Proposition 4.10]). It is also known that if π is relatively discrete series then the absolute values of all its generalized matrix coefficients belong to the Schwartz-Harish-Chandra space $\mathcal{C}(ZH \backslash G)$ (cf. [DH14, Lemma 4.2]).

3 H -integrability and very strongly discreteness

3.1 Supercuspidal representations

Now let π be a supercuspidal representation of G . As mentioned in the introduction, in such a case, we allow $\mathbf{X} = \mathbf{H} \backslash \mathbf{G}$ to be a wavefront spherical variety but require that \mathbf{G} is split. If this is the case, the definition of relatively supercuspidal is the same as the case of symmetric space. Since π is supercuspidal, the matrix coefficients of π are compactly supported modulo \mathbf{Z} and thus belong to $L^1(H/Z_H)$. Therefore the pairing \mathcal{L} is well defined. To prove Theorem 1.5, we need the following lemma, which is more or less well known. We present a proof for completeness.

Lemma 3.1. *If π is supercuspidal, it is relatively supercuspidal.*

Proof. When \mathbf{X} is a symmetric space, it is proved by Kato and Takano [KT08, Proposition 8.1].

When \mathbf{G} is split and \mathbf{X} is wavefront, the lemma follows from [SV12, Theorem 5.1.2] on asymptotics of the generalized matrix coefficients. We briefly explain the reason. Given $\ell \in \text{Hom}_H(\pi, \mathbb{C})$ and $v \in V_\pi$, we simply denote $f = \varphi_{\ell, v}$. Suppose that v is in V_π^J where J is an open compact subgroup of G and V_π^J is the subspace of V_π fixed by J . For each $\Theta \subset \Delta_X$ where Δ_X is the set

of spherical roots associated to \mathbf{X} (see [SV12, §2.1]), there is a boundary \mathbf{G} -spherical variety \mathbf{X}_Θ (see [SV12, §2.4]). Write $X_\Theta = \mathbf{X}_\Theta(F)$. The key fact is that, for $\Theta \subsetneq \Delta_X$, $C^\infty(X_\Theta)$ as a G -representation is parabolically induced from some P_Θ^- -representation where P_Θ^- is the parabolic subgroup of G associated to Θ . By [SV12, Theorem 5.1.2], for each $\Theta \subset \Delta_X$, there is a form $\varphi_\Theta \in \text{Hom}_G(\pi, C^\infty(X_\Theta))$ such that $f|_{N_\Theta} = \varphi_\Theta(v)|_{N'_\Theta}$ where N_Θ, N'_Θ are some “ J -good neighborhoods of Θ -infinity”. Since π is supercuspidal, we have $\varphi_\Theta(v) = 0$. Thus, $f|_{N_\Theta} = 0$ for each $\Theta \subsetneq \Delta_X$. At last, Lemma 3.1 follows from the property that the set

$$X \setminus \bigcup_{\Theta \subsetneq \Delta_X} N_\Theta$$

is compact modulo Z . \square

3.2 Very strongly discrete symmetric spaces

Let $\mathbf{H} \backslash \mathbf{G}$ be a symmetric space. Since the absolute values of matrix coefficients of discrete series representations belong to $\mathcal{C}(G/Z)$, the symmetric space $\mathbf{H} \backslash \mathbf{G}$ is strongly discrete if it is very strongly discrete. As shown in the proof of [Clo89, Lemma 1], $\mathbf{H} \backslash \mathbf{G}$ is very strongly discrete if and only if the following condition is satisfied:

(\star) there exists a natural number N such that

$$\int_{H/Z_H} \Xi(g)(1 + \sigma_*(g))^{-N} dh < +\infty. \quad (13)$$

As mentioned in Remark 1.10, a sufficient condition for strongly discreteness is obtained by Gurevich-Offen in [GO15, Corollary 5.4]. We will show that this condition is sufficient and necessary for very strongly discreteness (see Proposition 3.3). Before we explain the reason, we show some examples to illustrate the idea.

3.2.1 Galois pairs

Let \mathbf{H} be a connected reductive group over F and E a quadratic separable extension of F . Let $\mathbf{G} = \text{Res}_{E/F}(\mathbf{H}_E)$ be the Weil restriction of the base change of \mathbf{H} to E , and θ the involution on \mathbf{G} defined by the nontrivial Galois conjugation in $\text{Gal}(E/F)$. Then $\mathbf{H} = \mathbf{G}^\theta$. We say that (\mathbf{G}, \mathbf{H}) is a Galois pair with respect to E/F . The global theory of automorphic period on Galois pairs were studied in [LR03].

Proposition 3.2. *Let (\mathbf{G}, \mathbf{H}) be a Galois pair. Then the symmetric space $\mathbf{H} \backslash \mathbf{G}$ is very strongly discrete.*

Proof. The proof is analogous to that of [III10, Proposition 1.1]. For simplicity, without loss of generality, we assume that the center of \mathbf{H} is anisotropic.

Let $\mathbf{A}_{0,\mathbf{H}}$ be a maximal split torus of \mathbf{H} and $\mathbf{P}_{0,\mathbf{H}}$ a minimal parabolic subgroup of \mathbf{H} containing $\mathbf{A}_{0,\mathbf{H}}$. Then there exists a maximal split torus \mathbf{A}_0 of \mathbf{G} which is θ -stable such that $\mathbf{A}_{0,\mathbf{H}} \subset \mathbf{A}_0$ and a θ -stable parabolic subgroup \mathbf{P}_1 of \mathbf{G} such that $\mathbf{P}_{0,\mathbf{H}} = \mathbf{P}_1^\theta$. Let \mathbf{P}_0 be a minimal parabolic subgroup \mathbf{P}_0 such that $\mathbf{A}_0 \subset \mathbf{P}_0 \subset \mathbf{P}_1$. Let $A_{0,H}^+$ and A^+ be the subsets defined with respect to $P_{0,H}$ and P_0 respectively as in §2.

As explained in [GO15, §5.1], we have the relation

$$A_{0,H}^+ \subset A_0^+. \quad (14)$$

Let $\delta_{0,H}$, δ_0 and δ_1 be the modular characters of $P_{0,H}$, P_0 and P_1 respectively. Then we have the relation (cf. [LR03, Lemma 2.5.1])

$$\delta_1^{\frac{1}{2}}|_{P_{0,H}} = \delta_{0,H} \quad \delta_1|_{A_{0,H}} = \delta_0|_{A_{0,H}}. \quad (15)$$

Now let Ξ be the Harish-Chandra function of G defined with respect to P_0 and A_0 -good maximal compact subgroup K . Fix an $A_{0,H}$ -good maximal compact subgroup K_H of H . Then by Cartan decomposition (4) and relations (3), (5) and (7), to show (\star) , it suffices to show that there exists a natural number N such that the following series is convergent:

$$\sum_{a \in A_{0,H}^+ / A_{0,H}^1} \delta_{0,H}^{-1}(a) \delta_0^{\frac{1}{2}}(a) (1 + \sigma(a))^{-N}. \quad (16)$$

By the relation (15), such N does exist. \square

3.2.2 Other cases

For a general symmetric space $\mathbf{H} \backslash \mathbf{G}$, it is known that we can always find such a data: a maximal split torus $\mathbf{A}_{0,\mathbf{H}}$ of \mathbf{H} , a minimal parabolic subgroup $\mathbf{P}_{0,\mathbf{H}} \supset \mathbf{A}_{0,\mathbf{H}}$, a θ -stable maximal split torus $\mathbf{A}_0 \supset \mathbf{A}_{0,\mathbf{H}}$ of \mathbf{G} , a θ -stable parabolic subgroup \mathbf{P}_1 of \mathbf{G} such that $\mathbf{P}_{0,\mathbf{H}} = \mathbf{P}_1^\theta$, and a minimal parabolic subgroup \mathbf{P}_0 such that $\mathbf{A}_0 \subset \mathbf{P}_0 \subset \mathbf{P}_1$.

Suppose that the symmetric space $\mathbf{H} \backslash \mathbf{G}$ satisfies the following assumption

$$A_{0,H}^+ \subset A_0^+, \quad \text{and} \quad \delta_0^{1/2}|_{A_{0,H}^+} \geq \delta_{0,H}|_{A_{0,H}^+}. \quad (17)$$

Then, by the same arguments as in the case of Galois pairs, we can show that $\mathbf{H} \backslash \mathbf{G}$ is very strongly discrete. For example, the reader can check that the symmetric pair $(\mathrm{GL}_{2n}(F), \mathrm{GL}_n(E))$ satisfies the assumption (17).

3.2.3 General cases

In general the condition (17) is not always satisfied. Now we review some part of Gurevich-Offen's work [GO15].

Let W^H be the Weyl group of $A_{0,H}$ in H and $W^{H \backslash G}$ the Weyl group of $A_{0,H}$ in G . Then there is a natural embedding $W^H \subset W^{H \backslash G}$. By [GO15, Corollary 3.5.(3)], there is a particular set of representatives $[W^{H \backslash G} / W^H]$ for the coset $W^{H \backslash G} / W^H$. Set ρ_0^G be the usual half sum of positive roots of \mathbf{A}_0 with respect to \mathbf{P}_0 and ρ_0^H the usual half sum of positive roots of $\mathbf{A}_{0,\mathbf{H}}$ with respect to $\mathbf{P}_{0,\mathbf{H}}$. For $w \in [W^{H \backslash G} / W^H]$, set

$$\rho_{H \backslash G}^w = \rho_0^G - 2w(\rho_0^H),$$

which is viewed as an element in $\mathfrak{a}_{0,H}^*$ where $\mathfrak{a}_{0,H}^* = X^*(\mathbf{A}_{0,H}) \otimes_{\mathbb{Z}} \mathbb{R}$. Let $\Delta^{H \backslash G}$ be the set of non-zero restrictions to $\mathbf{A}_{0,\mathbf{H}}$ of the simple roots $\Delta(A_0, P_0)$ of \mathbf{A}_0 with respect to \mathbf{P}_0 . We say that $\rho_{H \backslash G}^w$ is relatively weakly positive if it is a linear combination of the elements of $\Delta^{H \backslash G}$ with non-negative coefficients.

Proposition 3.3. *The symmetric space $\mathbf{H} \backslash \mathbf{G}$ is very strongly discrete if and only if $\rho_{\mathbf{H} \backslash \mathbf{G}}^w$ is relatively weakly positive for every $w \in [W^{H \backslash G} / W^H]$.*

Proof. Set $A_{e,H}^+ = A_{0,H}^+ \cap A_0^+$. Let $\mathcal{N}^{H \backslash G}$ be a subset of $N_G(A_{0,H})$ consisting of a choice of a representative n for every element $w \in [W^{H \backslash G} / W^H]$. By [GO15, Corollary 3.5] there is a partition for $A_{0,H}^+$:

$$A_{0,H}^+ = \bigsqcup_{n \in \mathcal{N}^{G/H}} n^{-1} A_{e,H}^+ n.$$

Then, also by the same arguments as in the case of Galois pairs, we can show that the condition (\star) holds if and only if there exists $N \in \mathbb{N}$ such that the following series is convergent for each $n \in \mathcal{N}^{H \backslash G}$:

$$\sum_{a \in A_{e,H}^+ / (A_{e,H}^+ \cap A_{0,H}^1)} \delta_{0,H}^{-1}(n^{-1} a n) \delta_0^{\frac{1}{2}}(a) (1 + \sigma(a))^{-N},$$

which is equivalent to

$$\delta_0^{\frac{1}{2}}(\cdot)|_{A_{0,H}^+} \geq \delta_{0,H}(n^{-1} \cdot n)|_{A_{0,H}^+}$$

for each $n \in \mathcal{N}^{H \backslash G}$. The last condition is equivalent to $\rho_{\mathbf{H} \backslash \mathbf{G}}^w$ is relatively weakly positive for each $w \in [W^{H \backslash G} / W^H]$. \square

Remark 3.4. Gurevich-Offen prove that $\mathbf{H} \backslash \mathbf{G}$ is strongly discrete if $\rho_{\mathbf{H} \backslash \mathbf{G}}^w$ is relatively weakly positive for every $w \in [W^{H \backslash G} / W^H]$. They use this condition to show some special cases of symmetric spaces are strongly discrete. Therefore all the strongly discrete symmetric spaces in [GO15, §5] are very strongly discrete series.

4 Proof of Theorem 1.4

From now on, let $\mathbf{H} \backslash \mathbf{G}$ be a very strongly discrete symmetric space. For simplicity but without loss of generality, we assume that the center of \mathbf{G} is anisotropic.

Lemma 4.1. *Let \mathbf{A} be a θ -split torus of \mathbf{G} . Then as functions on $H \times A$, we have*

$$1 + \sigma(ha) \asymp 1 + \sigma(h) + \sigma(a).$$

Proof. Consider the natural map

$$p : \mathbf{H} \times \mathbf{A} \longrightarrow \mathbf{G}, \quad (h, a) \mapsto ha.$$

Let $\mathbf{T} = \mathbf{H} \cap \mathbf{A}$, which is a finite group. Then p is a composition of the quotient map

$$\tau : \mathbf{H} \times \mathbf{A} \rightarrow (\mathbf{H} \times \mathbf{A}) / \mathbf{T}$$

and the closed immersion

$$(\mathbf{H} \times \mathbf{A}) / \mathbf{T} \hookrightarrow \mathbf{G},$$

where the action of \mathbf{T} on $\mathbf{H} \times \mathbf{A}$ is $t \cdot (h, a) = (ht, t^{-1}a)$. Since τ is a finite morphism, the map p is also a finite morphism. Note that $\sigma(h) + \sigma(a)$ is a log-norm on $H \times A$ and $\sigma(ha)$ is the pullback by p of the log-norm σ on G . Therefore, by [Kot05, Proposition 18.1], as functions on $H \times A$, $1 + \sigma(ha)$ and $1 + \sigma(h) + \sigma(a)$ are equivalent. \square

Now let \mathbf{P} be a minimal θ -parabolic subgroup of \mathbf{G} . Denote $\mathbf{A} = \mathbf{A}_{\mathbf{P},\theta}$ and $A^+ = A_{P,\theta}^+$.

Lemma 4.2. *Suppose that there exists $N \in \mathbb{N}$ such that*

$$\int_H \Xi(h)(1 + \sigma(h))^{-N} \, dh$$

is convergent. Then there exists $d \in \mathbb{N}$ such that, as functions on A^+ , we have

$$\int_H \Xi(ha)(1 + \sigma(ha))^{-d} \, dh \prec \Xi(a).$$

Proof. By Lemma 4.1, we have for any $d \in \mathbb{N}$

$$(1 + \sigma(ha))^{-d} \prec (1 + \sigma(h) + \sigma(a))^{-d} \prec (1 + \sigma(h))^{-d}.$$

Let \bar{P} be the opposite of P . Let $C_1 \subset H, C_2 \subset \bar{P}$ be some compact neighborhoods of the identity. Since $H\bar{P}$ is open in G , there exists a compact neighborhood of the identity $C_K \subset K$ such that $C_K \subset C_1 C_2$. Note that there is a compact set C such that $a^{-1}C_2 a \subset C$ for any $a \in A^+$. Therefore there exists $d \in \mathbb{N}$ big enough such that for any $k = c_1 c_2 \in C_K$ with $c_1 \in C_1, c_2 \in C_2$, we have

$$\begin{aligned} \int_H \Xi(ha)(1 + \sigma(ha))^{-d} \, dh &\prec \int_H \Xi(ha)(1 + \sigma(h))^{-d} \, dh \\ &\prec \int_H \Xi(hc_1 a \cdot a^{-1}c_2 a)(1 + \sigma(hc_1))^{-N} \, dh \\ &= \int_H \Xi(hc_1 c_2 a)(1 + \sigma(h))^{-N} \, dh. \end{aligned}$$

Therefore, by (6), we have

$$\begin{aligned} \int_H \Xi(ha)(1 + \sigma(ha))^{-d} \, dh &\prec \int_H \int_{C_K} \Xi(hka)(1 + \sigma(h))^{-N} \, dk \, dh \\ &\prec \int_H \int_K \Xi(hka)(1 + \sigma(h))^{-N} \, dk \, dh \\ &= \Xi(a) \cdot \left(\int_H \Xi(h)(1 + \sigma(h))^{-N} \, dh \right), \end{aligned}$$

which completes the proof. \square

Lemma 4.3. *Let π be a discrete series representation of G . Then for any generalized matrix coefficient φ and matrix coefficient ϕ of π , the integral*

$$\int_G \varphi(g)\phi(g) \, dg$$

is absolutely convergent.

Proof. We have to show that the integral

$$\int_{H \setminus G} |\varphi(g)| \int_H |\phi(hg)| \, dh \, dg$$

is absolutely convergent. Consider the relative Cartan decomposition (9). Let $\mathbf{P} \in \mathcal{P}$ and $\mathbf{A} = \mathbf{A}_{\mathbf{P}, \theta}$. Choose a maximal split torus \mathbf{A}_0 of \mathbf{G} and a minimal parabolic subgroup \mathbf{P}_0 of \mathbf{G} such that $\mathbf{A} \subset \mathbf{A}_0 \subset \mathbf{P}_0 \subset \mathbf{P}$. It suffices to show that

$$\int_{H \backslash HA + \Omega} |\varphi(a)| \int_H |\phi(ha)| \, dh \, da \, dk$$

is absolutely convergent. Since π is discrete series, it is also relatively discrete series. Therefore the generalized matrix coefficient φ belongs to $\mathcal{C}(H \backslash G)$. Thus, for any $d \in \mathbb{N}$, we have

$$|\varphi(a)| \prec \Theta(a) \mathbf{N}_{-d}(a).$$

Since $\mathbf{H} \backslash \mathbf{G}$ is very strongly discrete and π is discrete series, by Lemma 4.2, for the inner integral we have

$$\int_H |\phi(ha)| \, dh \prec \Xi(a).$$

Hence it suffices to show that, for some $d \in \mathbb{N}$ big enough, the integral

$$\int_{H \backslash HA + \Omega} \Theta(a) \Xi(a) \mathbf{N}_{-d}(a) \, da \, d\gamma$$

is convergent. Note that $A^+ \subset A_0^+$. According to (5) and (11), there exists $d_1, d_2 \in \mathbb{N}$ such that

$$\Theta(a) \prec \delta_P^{\frac{1}{2}}(a) \mathbf{N}_{d_1}(a), \quad \Xi(a) \prec \delta_{P_0}^{\frac{1}{2}}(a) (1 + \sigma(a))^{d_2} = \delta_P^{\frac{1}{2}}(a) (1 + \sigma(a))^{d_2}.$$

The rest of the proof is the same as that of [DH14, Lemma 2.1]. \square

Let π be a discrete series representation of G . Now we define an hermitian inner product (\cdot, \cdot) on $\text{Hom}_H(\pi, \mathbb{C})$. Denote by $\bar{\pi}$ the complex conjugate of π . For $\ell_1, \ell_2 \in \text{Hom}_H(\pi, \mathbb{C})$ and $v, v' \in V_\pi$, the following integral is well defined:

$$\langle v, v' \rangle_{\ell_1, \ell_2} := \int_{H \backslash G} \varphi_{\ell_1, v}(g) \overline{\varphi_{\ell_2, v'}(g)} \, dg,$$

since the generalized matrix coefficients are square-integrable over $H \backslash G$. Then

$$v \otimes \bar{v}' \mapsto \langle v, v' \rangle_{\ell_1, \ell_2}$$

defines a morphism in $\text{Hom}_G(\pi \otimes \bar{\pi}, \mathbb{C})$. Since $\dim \text{Hom}_G(\pi \otimes \bar{\pi}, \mathbb{C}) = 1$, there exists $d_{\ell_1, \ell_2} \in \mathbb{C}$ such that

$$\langle v, v' \rangle_{\ell_1, \ell_2} = d_{\ell_1, \ell_2} \langle v, v' \rangle$$

for any $v, v' \in V_\pi$. Define

$$(\ell_1, \ell_2) = d_{\ell_1, \ell_2}. \tag{18}$$

It is obvious that (\cdot, \cdot) is an hermitian inner product on $\text{Hom}_H(\pi, \mathbb{C})$.

Proof of Theorem 1.4. Let $\mathcal{H}(\pi)^\perp$ be the orthogonal complement of $\mathcal{H}(\pi)$ in $\text{Hom}_H(\pi, \mathbb{C})$ with respect to the inner product (\cdot, \cdot) defined as (18). By [Del10, Theorem 4.5] we have

$$\dim \text{Hom}_H(\pi, \mathbb{C}) < \infty$$

for any irreducible admissible representation π . Hence, to show $\mathcal{H}(\pi) = \text{Hom}_H(\pi, \mathbb{C})$, it suffices to show that $\mathcal{H}(\pi)^\perp$ is zero.

Suppose that $\mathcal{H}(\pi)^\perp$ is nonzero and choose a nonzero element ℓ of $\mathcal{H}(\pi)^\perp$. We will show that there exists a vector v_0 in V_π so that

$$(\ell, \mathcal{L}_{v_0}) \neq 0,$$

which is a contradiction. For $u, v \in V_\pi$, set $\phi(g) = \langle \pi(g)v, u \rangle$ for the matrix coefficient associated to u, v . For $v_0 \in V_\pi$, set

$$I(v_0, \phi) = \int_G \phi(g) \overline{\varphi_{\ell, v_0}(g)} \, dg,$$

which is absolutely convergent by Lemma 4.3. Therefore

$$I(v_0, \phi) = \int_{H \backslash G} \varphi_{\mathcal{L}_{u, v}}(g) \overline{\varphi_{\ell, v_0}(g)} \, dg = \langle v, v_0 \rangle_{\mathcal{L}_{u, \ell}}.$$

On the other hand, $I(v_0, \phi)$ can be rewritten as

$$\lim_{n \rightarrow \infty} \int_{\Omega_n} \phi(g) \overline{\ell(\pi(g)v_0)} \, dg,$$

where Ω_n is an increasing family of open compact subsets of G and whose union is G . The integration over Ω_n is actually a finite sum. Thus ℓ can be moved out and we get

$$\begin{aligned} \int_{\Omega_n} \phi(g) \overline{\ell(\pi(g)v_0)} \, dg &= \overline{\ell \left(\int_{\Omega_n} \bar{\phi}(g) \pi(g)v_0 \, dg \right)} \\ &= \overline{\ell(\pi(\bar{\phi} \cdot \mathbf{1}_{\Omega_n})v_0)}, \end{aligned}$$

where $\mathbf{1}_{\Omega_n}$ is the characteristic function of Ω_n . Therefore, passage to $n \rightarrow \infty$, we obtain

$$\begin{aligned} &\lim_{n \rightarrow \infty} \int_{\Omega_n} \phi(g) \overline{\ell(\pi(g)v_0)} \, dg \\ &= \lim_{n \rightarrow \infty} \overline{\ell(\pi(\bar{\phi} \cdot \mathbf{1}_{\Omega_n})v_0)} \\ &= \overline{\ell(\pi(\bar{\phi})v_0)}. \end{aligned}$$

For the definition of $\pi(\phi)v_0$, we refer to [Wal03, §III.7].

Now we choose some specific $v_0 \in V_\pi$ so that $\ell(v_0) \neq 0$ and set $\phi_0(g) = \langle v_0, \pi(g)v_0 \rangle$. Then, by Schur orthogonality relations, $\pi(\phi_0)v_0 = \lambda v_0$ for some nonzero $\lambda \in \mathbb{C}$. Thus

$$\langle v_0, v_0 \rangle_{\mathcal{L}_{v_0, \ell}} = \overline{\ell(\pi(\phi_0)v_0)} = \overline{\lambda \ell(v_0)} \neq 0.$$

Therefore, $(\mathcal{L}_{v_0}, \ell) \neq 0$, which completes the proof. \square

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